

On the Stability of Einstein Static Universe in Doubly General Relativity Scenario

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Abstract

By presenting a relation between average energy of the ensemble of probe photons and energy density of the Universe, in the context of *gravity's rainbow* or *doubly general relativity* scenario, we introduce a rainbow FRW Universe model. By analyzing the fixed points in flat FRW model modified by two well known rainbow functions, we find that the finite time singularity avoidance (i.e. Big-Bang) may still remain as a problem. Then, we follow the “Emergent Universe” scenario in which there is no beginning of time and consequently there is no Big-Bang singularity. Moreover, we study the impact of a high energy quantum gravity modifications related to the gravity's rainbow on the stability conditions of an “Einstein static Universe” (ESU). We find that independent of a particular rainbow function, the positive energy condition dictates a positive spatial curvature for the Universe. In fact, without raising a nonphysical energy condition in the quantum gravity regimes, we can address an agreement between gravity's rainbow scenario and basic assumption of modern version of “Emergent Universe”. We show that in the absence and presence of an energy-dependent cosmological constant $\Lambda(\epsilon)$, a stable Einstein static solution is available versus the homogeneous and linear scalar perturbations under the variety of obtained conditions. Also, we explore the stability of ESU against the vector and tensor perturbations.

Keywords: Doubly General Relativity, Gravity's Rainbow, Einstein Static Universe, Stability Analysis.

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1 Introduction

In the framework of general relativity (GR), the gravitational force is explained in terms of the space-time curvature so that the field equations connect the space-time geometry to the matter content. More technically, GR exhibits a Universe modeled by space-time with a mathematical structure formed by four dimensional differentiable manifold [1]. It is commonly believed that a unique mathematical framework to reconcile the quantum mechanics (QM) with GR is highly dependent on our understanding of the space-time geometry. In other words, there is a comprehensive agreement that the geometry of space-time is fundamentally explained by a quantum theory. It is assumed that the Planck energy ϵ_{pl} addresses a critical energy scale of transition from classical GR to quantum gravity (QG), a theory which is supposed to challenge the most fundamental and long standing issues of modern physics. Despite the lack of a complete theory of QG, it seems a semi-classical or effective phenomenological approach of QG may guide us to disclose the mysterious nature of QG [2, 3, 4, 5]. It is important to understand how to extract the testable predictions from this fundamental theory. Interestingly, all semi-classical approaches offered so far unanimously insist on the existence of a minimal measurement length in the nature, known as the Planck length l_p . Besides, it is believed that the modified mass-shell condition (or dispersion relations) arising from deviation of Lorentz invariance in effective phenomenological approaches [4, 5], may justify some of the phenomena taking place in astronomical and cosmological scale, like the threshold anomalies of ultra high energy cosmic rays and TeV photons [6, 7]. It is noticeable that such Lorentz invariance violation is already predicted in the context of other approaches to QG such as: non-commutative geometry [7], spin network in Loop quantum gravity [8], and string field theory [9]. Throughout the present work, our attention specifically is focused on a semi-classical formalism known as “*Gravity’s Rainbow*” which has been designed by Magueijo and Smolin [10]. In fact, it can be said that the rainbow gravity is nothing but the doubly special relativity (DSR) [4] in the presence of curved space-time which is known as the “*Doubly General Relativity*” (DGR). In this formalism, there is no single fixed space-time background, namely the space-time background appears as a geometric spectrum in terms of the energy scale of the particle probe. Therefore, in the cosmological setting, we are dealing with the rainbow modified metric by introducing a function in the Friedmann-Robertson-Walker (FRW) metric which depends on such variable energy scale. Formation of the rainbow metric proposal involve various motivations such as the lack of a trivial definition of the dual position space in DSR [10]. However, this approach to QG is not problem free and considerable number of works have been focused on its various aspects [11]. Somehow, the proposal of gravity’s rainbow is similar to the idea of “*Running Coupling Constants*” in particle physics and field theory, in the sense that at very small distances (high energy) the space-time geometry is related to the energy scale of the particle probe [12]. It is worth mentioning that, apart from the effective approaches to QG like DSR and DGR, it has been argued that if the physical space-time of the standard GR theory is emergent, then it is expected that we face with a radical picture of the Universe at fundamental quantum scale [12]. The basic idea of “*Emergent Universe*” dates back to the works of Eddington in 1930 [13], inspired by the proposal of Einstein static Universe (ESU). In simple words, ESU suggests that the content of the Universe as a closed system is made up of at least cosmological constant and normal matter. While ESU in the context of GR is not stable versus spatially homogeneous perturbations, it predicts that the Universe in past might have been emerged as an static initial states [14]. We know that the data obtained by cosmic microwave background (CMB) observations, generally supports the inflationary scenario. Also, by implementation of some singularity theorems with the geometrical assumptions that $K < 0$ or $K = 0$, it has been shown that despite the occurrence of inflation, the Universe had a violent beginning in the past [15]. Hence, one of the most important motivations for studying QG is to avoid the singularity at the beginning of the Universe. Recovery of ESU in the framework of inflationary Universe by Ellis and Maartens, known as modern version of Eddington emergent Universe, was in this direction of research interest [16]. According to the modern standard cosmology, the inflationary expansion of the Universe has quickly eliminated any original spatial curvature. On the other hand, the CMB observations implies $\Omega_0 \gtrsim 1$ (1.02 ± 0.02) for the total density parameter. This means that the geometry of Universe is not exactly flat, rather it could have non-zero spatial curvature at the beginning, resulting in negligible late-time effects. In this sense,

the emergent Universe scenario allowed the existence of a positive curvature Universe that emanates asymptotically as a static initial state known as ESU which afterwards experienced the inflation. The stable ESU addresses the existence of a fixed point around which our Universe in the early times was eternally fluctuating. In the case of the Universe filled with a massless scalar field ϕ with a suitable inflaton potential, these fluctuations had been disappeared and the stable state of Universe had been turned to inflationary phase. Within the context of standard cosmology, the prerequisites of a one or further flat wings (of course with a little positive gradient) in the inflaton potential, can make an emergent Universe. This transmission from a stable ESU to inflationary phase takes place around the different values of e-folds. For instance, in [14] by analyzing the spectrum of inflationary perturbations of some stable cosmological models in the context of GR, it has been demonstrated that the transmission occurs in over 60 e-folds. Also, it should be noted that while these inflationary emergence GR-based models respect to CMB constraints, they suffer from a fine-tuning problem. This problem can be resolved in the context of modified emergence models of GR. However, recently in [17] some arguments are presented which tries to show the emergent cosmological scenario can not really be past-eternal. As a prominent feature of emergent Universe scenario, one can point to the removal of initial singularity and also the horizon problem. This is a strong motivation to study the ESU models along with their stability conditions in the presence of high energy corrections of GR. For instance, one can point to the works done in the context of massive gravity [18], Hořava-Lifshitz model of gravity [19], braneworld scenarios [20], induced matter theory [21], loop quantum cosmology [22], $f(R)$, $f(T)$ and $f(G)$ gravity [23]. As discussed at the beginning of this section, gravity's rainbow or DGR is also considered as an alternative of GR with high energy corrections. Besides, it has been shown that in the study of FRW cosmology and isotropic quantum cosmological perfect fluid model in rainbow gravity setup, some conditions are derived which prevent the initial singularity [25]. Inspired by this introduction, our main goal in this paper is to study the stability of ESU against the homogeneous scalar, vector and tensor perturbations in the presence of rainbow's gravity setup as a QG modification of GR. But before doing this, by following the phase space method we will analyze the status of the Big-Bang singularity in the framework of a flat rainbow FRW Universe model.

The present paper is arranged as follows: In section 2, we have derive rainbow modified Friedman equations in details for a general non-flat universe. In section 3, by introducing two different forms of rainbow functions depending on energy density, firstly in absence of cosmological constant, we obtain the stability conditions of ESU against the homogeneous linear scalar perturbations. In the follow of this section, we pursue our goal in the presence of a cosmological constant related to energy density via introducing a new rainbow function. In section 4, we analysis the stability against the vector and tensor perturbations. Finally in section 5, we give the summery and conclusions.

2 Gravity's Rainbow Modified FRW Universe

In this section, firstly we have a quick review on the gravity's rainbow theory, known also as DGR. Then, with note to the importance of spatial curvature in Emergent Universe scenario, we derive the modified Friedman equations for general non-flat universes. In DSR, the modified dispersion relations of a massive particle with mass m reads as

$$\epsilon^2 f^2(\epsilon/\epsilon_p, \eta) - p^2 g^2(\epsilon/\epsilon_p, \eta) = m^2, \quad (1)$$

where $f(\epsilon/\epsilon_p, \eta)$ and $g(\epsilon/\epsilon_p, \eta)$ are the well known energy dependent rainbow functions and η is a dimensionless parameter. In the low energy limit, the modified dispersion relation (1) reduces to the relativistic dispersion relations. So, the rainbow functions $f(\epsilon/\epsilon_p, \eta)$ and $g(\epsilon/\epsilon_p, \eta)$ satisfy the following conditions

$$\lim_{\epsilon/\epsilon_p \rightarrow 0} f(\epsilon/\epsilon_p, \eta) = 1, \quad \lim_{\epsilon/\epsilon_p \rightarrow 0} g(\epsilon/\epsilon_p, \eta) = 1. \quad (2)$$

Indeed, the condition (2) points to the correspondence principle. Based on the discussion carried out in [26], for established position space in DSR, theories including free fields must also lead to the plane

wave solutions in flat space-time, despite satisfying the modified dispersion relations (1). For this reason, contraction between infinitesimal displacement dx^a and momentum p_a , must be linearly invariant, i.e

$$dx^a p_a = dt\epsilon + dx^i p_i . \quad (3)$$

In fact, the linear contraction (3) guarantees the existence of the plane wave solutions. The rainbow metric generally has the form of

$$ds^2 = \frac{-1}{f^2(\epsilon/\epsilon_p, \eta)} dt^2 + \frac{1}{g^2(\epsilon/\epsilon_p, \eta)} dx^2 . \quad (4)$$

Using a one parameter family of energy momentum tensors, the gravity's rainbow modified Einstein equations will be written as

$$G_{\alpha\beta}(\epsilon/\epsilon_p) = 8\pi G(\epsilon/\epsilon_p) T_{\alpha\beta}(\epsilon/\epsilon_p) + g_{\alpha\beta}(\epsilon/\epsilon_p) \Lambda(\epsilon/\epsilon_p) , \quad (5)$$

so that $G(\epsilon/\epsilon_p) = h_1(\epsilon/\epsilon_p)G$ and $\Lambda(\epsilon/\epsilon_p) = h_2(\epsilon/\epsilon_p)\Lambda$ where $h_1(\epsilon/\epsilon_p)$ and $h_2(\epsilon/\epsilon_p)$ are energy dependent rainbow functions. This means that in DGR setup, the Newtonian gravitational constant and the cosmological constants are energy dependent such that in low energy limit, we recover $G(\epsilon/\epsilon_p) = G$ and $\Lambda(\epsilon/\epsilon_p) = \Lambda$. In order to achieve our main goal, in this section we will set $\Lambda = 0$ for simplicity, and begin with the following modified FRW metric of a homogeneous and isotropic Universe

$$ds^2 = -\frac{N(t)^2}{f^2(\epsilon/\epsilon_p, \eta)} dt^2 + \frac{1}{g^2(\epsilon/\epsilon_p, \eta)} a^2(t) h_{ij} dx^i dx^j , \quad (6)$$

where h_{ij} represents the spatial part of the metric. Commonly, it is assumed that the energy ϵ of particle probe for each measurement is constant and independent of space-time coordinates. Nevertheless, such assumption for the measurements at early Universe seems to be far from reality. Therefore, it is expected that the background metric of space-time throughout its evolution is affected by the energy ϵ of particle probe. Then, it is reasonable to consider the evolution of energy ϵ with the cosmological time, denoted as $\epsilon(t)$. In this regard, we will derive the modified FRW equations. In doing so, the following ansatz is usually suggested [27]

$$\eta = 1, \quad g^2(\epsilon, \eta) = 1 , \quad (7)$$

where f has the form of $f(\epsilon/\epsilon_p)$. Therefore, the modified Einstein field equation can be written as

$$R_{\alpha\beta} = -8\pi G(\epsilon) S_{\alpha\beta}(\epsilon) , \quad (8)$$

where $R_{\alpha\beta}$ is Ricci tensor defined as follows

$$R_{\alpha\beta} = \frac{\partial \Gamma_{\lambda\alpha}^{\lambda}}{\partial x^{\beta}} - \frac{\partial \Gamma_{\alpha\beta}^{\lambda}}{\partial x^{\lambda}} + \Gamma_{\alpha\mu}^{\lambda} \Gamma_{\beta\lambda}^{\mu} - \Gamma_{\alpha\beta}^{\lambda} \Gamma_{\lambda\mu}^{\mu}, \quad \alpha, \beta = 0, 1, 2, 3, \quad (9)$$

and $S_{\alpha\beta}(\epsilon)$ is written in terms of the energy momentum tensor $T_{\alpha\beta}(\epsilon)$ as

$$S_{\alpha\beta}(\epsilon) = T_{\alpha\beta}(\epsilon) - \frac{1}{2} g_{\alpha\beta} T_{\mu}^{\mu}(\epsilon) . \quad (10)$$

To continue our calculations, we need the non-zero components of the affine connection as [27]

$$\Gamma_{00}^0 = -\frac{\dot{f}}{f}, \quad \Gamma_{ij}^0 = f^2 \dot{a} a \delta_{ij}, \quad \Gamma_{0j}^i = \delta_j^i \frac{\dot{a}}{a} \quad i, j = 1, 2, 3 . \quad (11)$$

It is seen that unlike the usual FRW metric, for modified FRW metric (6), the components of the affine connection with two time indices remain non-zero. By putting (11) into (9) and after a straightforward calculation, we obtain

$$R_{00} = 3 \frac{\ddot{a}}{a} + 3 \frac{\dot{a}}{a} \frac{\dot{f}}{f} , \quad (12)$$

and

$$R_{ij} = \tilde{R}_{ij} - \left((a\ddot{a} + 2\dot{a}^2)f^2 - f\dot{f}\dot{a} \right) \delta_{ij} , \quad (13)$$

where \tilde{R}_{ij} denotes the purely spatial Ricci tensor defined as

$$\tilde{R}_{ij} = \frac{\partial \Gamma_{ni}^n}{\partial x^j} - \frac{\partial \Gamma_{ij}^n}{\partial x^n} + \Gamma_{in}^m \Gamma_{jm}^n - \Gamma_{ij}^m \Gamma_{nl}^n . \quad (14)$$

The spatial components Γ_{jk}^i of the four dimensional affine connection are identical with those of affine connection computed in three dimensions from the spatial three-metric h_{ij} , i.e.

$$\Gamma_{jk}^i = \frac{1}{2} h^{in} \left(\frac{\partial h_{jn}}{\partial x^k} + \frac{\partial h_{kn}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^n} \right) \equiv \tilde{\Gamma}_{jk}^i , \quad (15)$$

where $\tilde{\Gamma}_{jk}^i$ denotes purely spatial affine connection and h^{ij} is the inverse of the 3×3 matrix h_{ij} . Therefore, we have $\Gamma_{ij}^n = kx^n h_{ij}$ which results in

$$\tilde{R}_{ij} = -2kh_{ij} . \quad (16)$$

Here, k has a geometrical interpretation. Indeed, it measures the spatial curvature with zero, negative and positive k values corresponding to flat, open and closed Universes respectively [28]. The Ricci tensor expression (13) can be rewritten as

$$R_{ij} = - \left((a\ddot{a} + 2\dot{a}^2)f^2 - f\dot{f}\dot{a} + 2k \right) h_{ij} . \quad (17)$$

Now, for obtaining the components of $S_{\alpha\beta}$, we consider the perfect fluid with the energy-momentum tensor as

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + pg_{\alpha\beta} , \quad (18)$$

so that ρ and p represent the energy density and the pressure, respectively. Also, u_α is the velocity four vector defined by $u_\alpha = (f^{-1}, 0, 0, 0)$, with the norm of $g^{\alpha\beta}u_\alpha u_\beta = -1$. We should note that the diagonal components of the modified FRW metric tensor $g_{\alpha\beta}$ is $(-f^{-2}, h_{ii})$ with the signature $(-, +, +, +)$ so that the components of spatial tensor h_{ii} are independent of the rainbow function f . Using the equation (10), one can decompose the time and spatial components of $S_{\alpha\beta}$ tensor as follows

$$S_{tt} = T_{tt} + \frac{1}{2}g_{tt}T, \quad S_{ij} = T_{ij} - \frac{1}{2}h_{ij}a^2T , \quad (19)$$

where

$$T_{tt} = \rho f^{-2}, \quad T_{ij} = h_{ij}a^2p , \quad (20)$$

and

$$T = g^{\alpha\beta}T_{\alpha\beta} = (-\rho + 3p) . \quad (21)$$

By substituting expressions (20) and (21) into (19), we have

$$S_{tt} = \frac{1}{2f^2}(\rho + 3p), \quad S_{ij} = \frac{1}{2}(\rho - p)a^2h_{ij} . \quad (22)$$

Ultimately, the first and second rainbow modified Friedman equations for the metric parameterized by the varying energy probe (6), takes the form of

$$\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \frac{1}{f^2} = \frac{8\pi G\rho}{3} \frac{1}{f^2} , \quad (23)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho(1+3\omega)}{3} \frac{1}{f^2} - \frac{\dot{a}}{a} \frac{\dot{f}}{f} . \quad (24)$$

Here, also for simplicity, it is assumed that gravitational constant G is independent of energy ϵ . It should be noted that the existence of f term in the second rainbow modified Friedman equations, does not mean the explicit dependence of rainbow function to the time i.e. $f(t)$, at all. As explained above, f is an explicit function of the energy of the test particles which are probing the geometry of space-time in early Universe. Given the fact that ϵ can vary with respect to the evolution of Universe, so f can be an implicit function of cosmic time and not explicit. We note that, the explicit dependence of f to the time, may be leads to this wrong result from equations (23) and (24) that they are nothing but the usual Friedman equations of GR so that the rainbow function f in this case is merely the gauge parameter determining the choice of time. Hence, based on the gauge freedom, one may choose the gauge of $f = 1$. In contrast to this misconception, the gravity's rainbow theory is a high energy modified theory of GR in which according to *correspondence principle*, for the case of low energy limit i.e. $\frac{\epsilon}{\epsilon_{Pl}} \rightarrow 0$ ($f \rightarrow 1$), modified Friedman equations (23) and (24) reduce to the standard equations for the FRW universe. Also, we mention that due to the existence of the rainbow functions, t may play the role of proper time based on the gauge choice $N = \frac{f_1(\epsilon)}{f_2^{3\omega}(\epsilon)} a^{3\omega}$ [25]. As expected, in the limit of GR, t represents the proper time indicated by the choice of $N = a^{3\omega}$. Also, by combining the modified Friedman equations (23) and (24), we get the following energy conservation equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1 + \omega) = 0. \quad (25)$$

It can be mentioned that in framework of DGR, the form of equation of state (EOS) $p = \omega\rho$, remains unchanged for massless prob particles such as photons while for the massive ones, this equation of state will be modified, see [29, 10] for more discussion.

3 Einstein Static Universe and Scalar Perturbations

3.1 In the Absence of Cosmological Constant Λ

In this section, we plan to apply the linear homogeneous scalar perturbations in the vicinity of the Einstein static Universe and explore its stability against these perturbations. To achieve our goal, we need to fix the rainbow function f . To this end, using dispersion relation offered in [30], we introduce rainbow functions as follows

$$f(\epsilon) = \left(1 - \frac{\epsilon}{\epsilon_{Pl}}\right)^{-1}, \quad (26)$$

where via suggesting the average energy $\bar{\epsilon} = \frac{4c}{3}\rho^{\frac{1}{4}}$ (c is some constant) [31], this rainbow function will be rewritten as

$$f(\rho) = \left(1 - \frac{4}{3}\xi\rho^{\frac{1}{4}}\right)^{-1}, \quad (27)$$

where $\xi = \frac{c}{\epsilon_{Pl}}$. The authors of [31], to reach the above relation considered a large ensemble of prob photons which are in thermal equilibrium. This assumption is reasonable since based on the standard model of cosmology, early universe passed a radiation dominated era with $p = \frac{1}{3}\rho$. One may ask the question that why the identification $\epsilon \sim \bar{\epsilon}$ is used to obtain the rainbow function (27)? The answer is that, we are deal with the energy of prob particle in the rainbow metric as the statistical mean value of all prob photons in radiation domination. Indeed, we deal with the average effect of photon particles in radiation dominated era and no with a special elected photon from the radiation [27]. It is worthwhile to remind that the form of $\bar{\epsilon}$ in terms of ρ is independent of the modified dispersion relation picked for a specific model [24]. We also mention that the above rainbow function is valuable from the theoretical viewpoint so that in [30] it is shown that in the absence of the varying speed of light (VSL) proposal, it can removes the horizon problem in early Universe. Moreover, it is seen that when ρ or $\bar{\epsilon}$ get their largest values, then $f(\rho)$ or $f(\bar{\epsilon})$ becomes infinite which causes the time-like component of the rainbow metric vanishing. This means that the FRW metric which is modified with rainbow function (27) becomes degenerate at that time and has no inverse. Indeed, a degenerate metric addresses the existence of

other distinct possibility for lightlike dimension (other than timelike and spacelike dimensions). Also, remember that in the Palatini formulation of standard GR, degenerate metric also appears. One of the consequences the degenerate metric is that the curvature remains bounded and the topology of space-time can change. Overall, it is believed that singularities arisen from degenerate metric have a milder manner than other type of singularities and seem appropriate for a QG proposal [32]. Here, it is necessary to review the finite time singularity issue, including the Big-Bang singularity, in the presence of rainbow function (27). Indeed, we want to investigate whether the presence of rainbow function (27) in this system will lead to resolve the singularity problem. Inspired by the idea proposed in [33, 31], we want to resolve this problem by finding an upper bound on the density of energy ρ . In other words, according to [33, 31] the finite time singularity issue will be eliminated via the presentation of a fixed point as ρ_f for energy density ρ which is reached at an infinite time. More exactly, the author of [31], by following the terminology used for stability analyzing the dynamical systems in [33], demonstrated that for any first order system as $\dot{\rho} = O(\rho)$, the finite time singularity will be solved through the fulfillment of either of the following conditions: 1) the function $O(\rho)$ be a continuous and differentiable on a range enclosed by zeroes of function $O(\rho)$. 2) asymptotically, function $O(\rho)$ grows like a linear function as $K(\rho)$ or slower than it, i.e $K(\rho) \geq O(\rho)$. Therefore, for cancelation of the finite time singularity, it is sufficient that one of these conditions is satisfied. We begin our analysis in this way by substituting the rainbow function (27) into energy conservation equation (25) and modified first Friedmann equation (23) (for case $k = 0$) to obtain the ordinary differential equation (ODE) as $\dot{\rho} = O(\rho)$ where

$$O(\rho) = -4\rho(1 - \xi\rho^{\frac{1}{4}}) \left(\frac{8\pi G}{3}\rho \right)^{1/2}. \quad (28)$$

The advantage of the dynamical system analysis method followed in [33, 31] is that by having fixed points and regarding the asymptotic behavior of $O(\rho)$, one is able to predict the demeanor of the system without need to have a detailed form of the solutions. Hence, by setting $\omega = \frac{1}{3}$ (since our attention is on radiation dominated state in order to study the initial singularity), equation (28) results in the following two fixed points

$$\rho_{f_1} = 0, \quad \rho_{f_2} = \left(\frac{\epsilon_{Pl}}{c} \right)^4. \quad (29)$$

However, to understand the qualitative manner of a solution, we should know that how long it takes to get a fixed point. By a straightforward calculation, one can show that the time required to reach these fixed points, can be obtained as

$$t = - \int_{\rho^*}^{\rho_f} \frac{d\rho}{\left(\frac{8\pi G}{3}\rho \right)^{1/2} \left(4\rho - \frac{16\xi}{3}\rho^{5/4} \right)}, \quad (30)$$

where ρ^* represents an arbitrary initial finite value for density which lies in the intervals between the fixed points. The negative sign in the back of integral relation, denotes a backward in direction of time. We find that for the case of $\rho_{f_1} = 0$, the integral (30) does not converge i.e. $|t| \rightarrow \infty$, while for the fixed point $\rho_{f_2} = \left(\frac{\epsilon_{Pl}}{c} \right)^4$ the integral (30) is not solvable and its numerical values fail to converge. At this point, let us to draw the plot $O(\rho) - \rho$ (or $\dot{\rho} - \rho$) in the figure 1, in order to get a qualitative analysis of the situation. As it is seen from the figure 1, $O(\rho)$ has two fixed points $\rho_{f_1} = 0$ and $\rho_{f_2} = 1$. In fact, any of these fixed points are equivalent to a de Sitter space since in these points we are dealing with constant solution ($\dot{\rho} = 0$). The first ρ_{f_1} is a future fixed point because for the case $\rho^* > 0$ we have $O(\rho) < 0$, while the second ρ_{f_2} is an early (or past) fixed point since for case of $\rho^* < 1$ and $\rho^* > 1$ we have $O(\rho) < 0$ and $O(\rho) > 0$, respectively. It is noteworthy that for distinction between the type of this singularities, we follow Ref. [31]. These fixed points classify the possible solutions into two classes: 1) a solution belongs to interval $\rho \in [0, 1]$, if $\rho_{f_1} < \rho^* < \rho_{f_2}$ and 2) a solution belongs to interval $\rho \in [1, \infty)$, if $\rho^* > \rho_{f_2}$. Then, to get the fixed point ρ_{f_1} by beginning from some initial value as ρ^* and also from ρ_{f_2} to ρ^* , the time required is infinite. In this interval, $O(\rho)$ is continuous and differentiable. Then, according to approach in [31] by pursuing the terminology of [33], one can say that the first solution

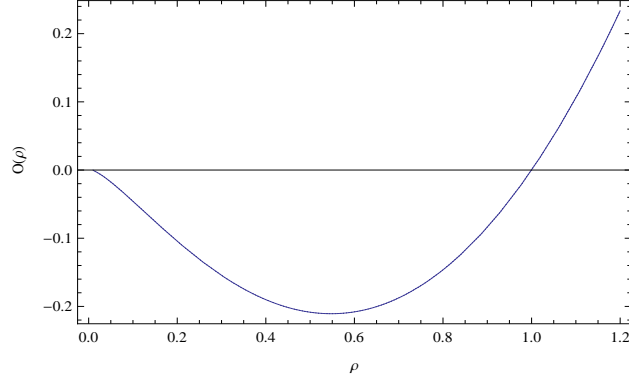


Figure 1: The behavior of $O(\rho)$ (28) in terms of ρ for flat rainbow FRW Universe model. To simplify, we set $\epsilon_{pl} = c$ and $8\pi G = 3$. Fixed points are located in $\rho_{f_1} = 0$ and $\rho_{f_2} = 1$.

is free of physical singularity so that it interpolates steadily from ρ_{f_1} to ρ_{f_2} . It is clear that the first condition already is not usable for the second solution, which represents a clear violation of the second condition. On the other hand, for the case $\rho^* > \rho_{f_2}$, the function (28) is growing quicker than a linear function. So, we conclude that the second solution is not free of a finite time singularity. Because we are interested in studying initial singularity problem, so let us mention to stability status of early fixed point ρ_{f_2} . A fixed point is stable if by putting a neighboring initial value, the trajectory of the solution remains always near to the fixed point. Equivalently, a fixed point is unstable if for any point at vicinity of the fixed point, one can find some solution that starts near the fixed point but go away from it in a finite time. The fixed point ρ_{f_2} is an unstable point because $\frac{dO(\rho)}{d\rho}|_{\rho=\rho_{f_2}} > 0$ ¹. Then, although it takes an infinite time to get a fixed point, the issue of finite time singularities avoidance remains unsolved and eventually this fixed point will collapse.

Now, we investigate the DGR theory form the emergent universe point of view in which there is no beginning of time and consequently, there is no big bang singularity in the early Universe. In this scenario, the Universe did not born from a Big-Bang singularity in a past finite time and rather it possesses an eternal Einstein static state. In this context, the key point is the required conditions for the stability of the existing ESU, in which we will explore in the following of the present paper. The ESU in the DGR scenario with rainbow functions varying with cosmological time can be obtained by the conditions $\ddot{a} = \dot{a} = 0$ through the equations (23) and (24) as

$$\frac{k}{a_0^2} = \frac{8\pi G\rho_0}{3}, \quad (31)$$

and

$$-\frac{4\pi G\rho_0(1+3\omega)}{3f(\rho_0)^2} = 0, \quad (32)$$

where a_0 , ρ_0 and ω denote the scale factor, the energy density of the Einstein static Universe and barotropic equation of state parameter ($p = \omega\rho$), respectively. Considering the positive energy condition $\rho_0 > 0$ through the equation (31), it is seen that for the Einstein static Universe, the spatial curvature of the Universe should be positive, $k > 0$. Also, Eq.(32), shows that for the Einstein static Universe in the

¹At a fixed point where $O(\rho_{f_2}) = 0$, if $\frac{dO(\rho)}{d\rho}|_{\rho=\rho_{f_2}} > 0$, we have increasing O at ρ_{f_2} , or equivalently $f(\rho_{f_2} - \delta) < 0 < f(\rho_{f_2} + \delta)$ for all sufficiently small and positive δ . This means that if we start with initial value $\rho^* > \rho_{f_2}$ in the vicinity of ρ_{f_2} , since $f(\rho^*) > 0$ then the trajectory of the ODE solution increases its value of ρ_{f_2} and moves away from the fixed point. Also if we start with $\rho^* < \rho_{f_2}$, but near to ρ_{f_2} , again trajectory of the ODE solution moves away from the fixed point but by decreasing its value of ρ_{f_2} . Therefore, if $\frac{dO(\rho)}{d\rho}|_{\rho=\rho_{f_2}} > 0$, we conclude that the fixed point ρ_{f_2} is unstable. Similarly, one can show that if $\frac{dO(\rho)}{d\rho}|_{\rho=\rho_{f_2}} < 0$, the fixed point will be stable.

framework of rainbow gravity, we need to have $\omega = -\frac{1}{3}$ or $f(\rho_0) \rightarrow \infty$ corresponding to $\rho_0 = (\frac{4}{3}\xi)^{-4}$. The perturbations in the cosmic scale factor $a(t)$ and the energy density $\rho(t)$ can be written as

$$\begin{aligned} a(t) &\rightarrow a_0(1 + \delta a(t)), \\ \rho(t) &\rightarrow \rho_0(1 + \delta \rho(t)). \end{aligned} \quad (33)$$

Substituting (33) into the equation (23) after linearizing the perturbation terms, we obtain the following equation

$$-\frac{2k}{a_0^2}\delta a = \frac{8\pi G}{3}\rho_0\delta\rho. \quad (34)$$

This indicates, with respect to the positiveness of k , that the sign of variation of the scale factor must be opposite to the sign of variation of the matter density. We must also apply the same method on the equation (24), however before doing this let us introduce the following replacement due to the perturbation in the rainbow function $f(\rho)$

$$f^{-2}(\rho) \rightarrow \left(f^{-2}(\rho_0) - \frac{2}{3}\xi\rho_0^{\frac{1}{4}}\delta\rho \right), \quad (35)$$

such that

$$f^{-2}(\rho_0) = 1 - \frac{8}{3}\xi\rho_0^{\frac{1}{4}}. \quad (36)$$

Also, by combining the energy conservation equation (25) with the first rainbow modified FRW equation (23), we obtain

$$-\frac{\dot{a}}{a}\frac{\dot{f}}{f} = -\frac{\dot{a}}{a}\frac{df}{d\rho}\frac{\dot{\rho}}{f} = \frac{3(1+3\omega)}{4}\xi f^{-1}\rho^{\frac{1}{4}}\left(\frac{8\pi G}{3}\rho - \frac{k}{a^2}\right). \quad (37)$$

Then, after applying perturbation (33), we have

$$-\frac{\dot{a}}{a}\frac{\dot{f}}{f} = \frac{3(1+3\omega)}{4}\xi\rho_0^{\frac{1}{4}}\left(1 + \frac{1}{4}\delta\rho\right)\left(f^{-1}(\rho_0) - \frac{\xi}{3}\rho_0^{\frac{1}{4}}\delta\rho\right)\left(\frac{8\pi G}{3}\rho_0(1 + \delta\rho) - \frac{k}{a_0^2}(1 - 2\delta a)\right). \quad (38)$$

By substituting (31) and (34) into the above expression, one finds that $-\frac{\dot{a}}{a}\frac{\dot{f}}{f} = 0$. It seems that this result is independent of any specific form of rainbow function f . Therefore, it can be seen that the last term of the second rainbow modified FRW equation (24) does not contribute in derivation of stability conditions of Einstein static Universe in the framework of DGR scenario. Now, putting perturbation equations (33) into (24) and using (35), we get

$$\delta\ddot{a} = -\frac{4\pi G}{3}(1+3\omega)\left(f_0^{-2}(\rho_0) - \frac{2}{3}\xi\rho_0^{\frac{1}{4}}\right)\rho_0\delta\rho, \quad (39)$$

where inserting the equation (34) into (39) and neglecting the non-linear perturbation terms, results in the following differential equation

$$\delta\ddot{a} - \frac{k}{a_0^2}(1+3\omega)\left(1 - \frac{10}{3}\xi\rho_0^{\frac{1}{4}}\right)\delta a = 0. \quad (40)$$

It is obvious that, for the cases $\xi \rightarrow 0$, i.e. $\epsilon_{Pl} \rightarrow \infty$, Eq.(40) will take the following form

$$\delta\ddot{a} - \frac{k}{a_0^2}(1+3\omega)\delta a = 0, \quad (41)$$

which refers to the oscillatory modes of ESU in the framework of standard GR for $\omega < -1/3$. In order to have the general oscillating perturbation modes in the framework of the DGR scenario, the following condition should be satisfied

$$-\frac{k}{a_0^2}(1+3\omega)\left(1 - \frac{10}{3}\xi\rho_0^{\frac{1}{4}}\right) > 0, \quad (42)$$

which results in the following solution for the equation (40)

$$\delta a = \alpha_1 e^{i\gamma_0 t} + \alpha_2 e^{-i\gamma_0 t}, \quad (43)$$

where α_1 and α_2 are integration constants and γ_0 refers to the frequency of oscillation around the stable ESU as

$$\gamma_0 = \left(-\frac{k}{a_0^2} (1 + 3\omega) \left(1 - \frac{10}{3} \xi \rho_0^{\frac{1}{4}} \right) \right)^{\frac{1}{2}} \quad (44)$$

Now, we can analysis the stability condition (42). Eqs.(31) and (32) will play the role of two fundamental constraints to achieve our goal. There are two possibility in order to satisfy the equation (32); the first one is that $\omega = -\frac{1}{3}$ and the second one is that the matter density of the ESU be $\rho_0 = \left(\frac{4}{3}\xi\right)^{-4}$. If we set $\omega = -\frac{1}{3}$, then the condition (42) is automatically violated and we obtain $\delta\ddot{a} = 0$ implying that there is no stable ESU against the linear scalar perturbations in the form of relations (33). As it is clear from Eq.(41), in the framework of the standard GR for $\omega = -\frac{1}{3}$, there is no stable ESU. For the second possibility as $\rho_0 = \left(\frac{4}{3}\xi\right)^{-4}$, with respect to the positivity of the spatial curvature, due to the positive energy condition through Equation (31), we need that the barotropic EOS parameter satisfies $\omega > -1/3$. Overall, one can find that there is a stable ESU versus homogenous and linear scalar perturbations in context of the gravity's rainbow scenario with choosing the rainbow function (27). In this case, we need two bounds which the first one in on EOS parameter as $\omega > -1/3$ and the second one is on the energy density of ESU as $\rho_0 < \epsilon_{pl}^4$. There are no such similar results in the framework of the standard GR theory. Unlike GR, here there is a stable solution against small scalar perturbations for a closed universe filled by the matter fields respecting the energy conditions. Also, within standard GR theory and even in many of the modified gravity theories, there is no such upper bounds on energy density of ESU ρ_0 . In the following, we want to investigate the stability conditions of Einstein static Universe against the linear homogeneous scalar perturbations in the context of DGR, in terms of another rainbow function proposed by Ling and et al in [35], which is evolving with cosmic time as

$$f = \sqrt{1 - l_p^2 \bar{\epsilon}^2}. \quad (45)$$

Considering previous arguments after placing the average energy $\bar{\epsilon} = \frac{4}{3}c\rho^{\frac{1}{4}}$ [31], it can be rewritten in terms of energy density ρ as

$$f(\rho) = \sqrt{1 - \frac{16}{9}\chi\rho^{\frac{1}{2}}}, \quad (46)$$

where $\chi = l_p^2 c^2$ is a constant. Let us point out that rainbow function (46) has application in the framework of black hole physics leading to valuable phenomenological outcomes, e.g. see [35, 36]. Moreover, it also as previous rainbow function has a theoretical trait. By reaching the energy density ρ or $\bar{\epsilon}$ to its largest value, $f(\rho)$ in (46) becomes zero which leads to vanishing spatial-like components of the rainbow metric. This means that FRW metric modified by (46) or (45), is a smooth and differentiable metric but is not invertible. Moreover, by applying the rainbow functions (45) or (46), we will be faced with a degenerate metric at energy levels close to the Planck energy scale. Also, based on the fact that there is no observational evidence for lightlike dimension in nature, there might be a suppression mechanism for appearance of lightlike dimension at the quantum level of Universe [32].

As before, we first examine the Big-Bang singularity problem for such a choice of rainbow function within a flat rainbow FRW model. In the same way, for the rainbow function (46), we get

$$\dot{\rho} = -4\sqrt{\frac{8\pi G\rho^3}{3 - \frac{16}{3}\chi\rho^{\frac{1}{2}}}}, \quad (47)$$

which has only one fixed point $\rho_{f_1} = 0$ and is not bounded, see Figure 2. As it is seen from the figure, the function $O(\rho)$ begins from the fixed point ρ_{f_1} and ends with a singularity. So we must separately

calculate the time needed from an infinite to a finite value ρ^* as well as the time needed from ρ^* to the fixed point $\rho_{f_1} = 0$, i.e,

$$t = -\frac{1}{4} \int_{\infty}^{\rho^*} d\rho \sqrt{\frac{3 - \frac{16}{3}\chi\rho^{\frac{1}{2}}}{8\pi G\rho^3}}, \quad (48)$$

and

$$t = -\frac{1}{4} \int_{\rho^*}^0 d\rho \sqrt{\frac{3 - \frac{16}{3}\chi\rho^{\frac{1}{2}}}{8\pi G\rho^3}}. \quad (49)$$

Here, we are dealing with a different situation compared to the previous one. We found that the integral (48) converges which provide $\rho^* \geq \frac{81}{256\chi^2}$ while the integral (49) does not converge anyway on the given interval. This means that from ρ^* to fixed point $\rho_{f_1} = 0$ an infinite time is needed, i.e $|t| \rightarrow \infty$, while to get an infinite to a finite value ρ^* it takes a finite time. Because the rainbow FRW metric includes a natural cutoff as the energy Planck, the constraint obtained for integral (48) should lie in the interval $\frac{81}{256\chi^2} \leq \rho^* \leq \frac{1}{\chi^2}$. Then, the solution derived from the rainbow function (46) for the flat rainbow Universe model can not be free of finite time singularity. We conclude that even assuming that integral

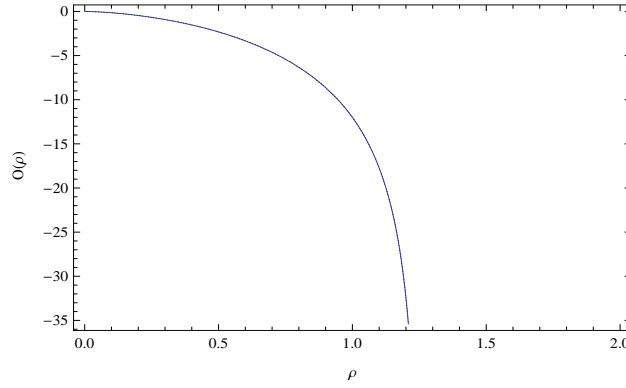


Figure 2: The behavior of $O(\rho)$ (47) in terms of ρ for the flat rainbow FRW Universe model. To simplify, we set $8\pi G = 3$ and $\chi = 1/2$.

(48) results in an infinite time, the Big-Bang singularity issue is not canceled, yet. Because according to the discussion given in the previous case, one can realize that ρ_{f_1} is a future fixed point and is not a past one.

In the following, we study the DGR theory from the emergent universe point of view and investigate the required conditions for the stability of an ESU with respect to scalar perturbations, in the presence of rainbow function (46). By applying the scalar perturbation terms on the rainbow function (46), it takes the form

$$f^{-2}(\rho) \rightarrow f^{-2}(\rho_0) + \frac{8}{9}\chi\rho_0^{\frac{1}{2}}\delta\rho, \quad (50)$$

where

$$f^{-2}(\rho_0) = 1 + \frac{16}{9}\chi\rho_0^{\frac{1}{2}}. \quad (51)$$

Now, inserting the equations (50), (51) and the perturbation equations (33) into the second rainbow modified Friedmann equation (24), we have

$$\delta\ddot{a} = -\frac{4\pi G}{3}(1+3\omega) \left((\rho_0 + 1)f^{-2}(\rho_0) + \frac{8}{9}\chi\rho_0^{\frac{1}{2}} \right) \rho_0\delta\rho, \quad (52)$$

where similar to the previous analysis, we neglected non-linear terms. Finally, by applying the equations (32) and (34), the above differential equation takes the following form

$$\delta\ddot{a} - \frac{k}{a_0^2}(1+3\omega) \left(1 + \frac{8}{3}\chi\rho_0^{\frac{1}{2}} \right) \delta a = 0. \quad (53)$$

As can be seen, for the limit $\chi \rightarrow 0$, the above equation reduces to the oscillatory modes (41). Then, in order to have a stable ESU in the framework of DGR scenario, the following condition should be satisfied

$$-\frac{k}{a_0^2}(1+3\omega)\left(1+\frac{8}{3}\chi\rho_0^{\frac{1}{2}}\right) > 0, \quad (54)$$

where the frequency of oscillatory modes γ_0 for the rainbow function (46) reads as

$$\gamma_0^2 = -\frac{k}{a_0^2}(1+3\omega)\left(1+\frac{8}{3}\chi\rho_0^{\frac{1}{2}}\right). \quad (55)$$

It is seen that for $\omega = -\frac{1}{3}$, the condition (54) is automatically violated and we get $\delta\ddot{a} = 0$ through the equation (53), which indicates that there is no stable ESU against the linear scalar perturbations for this rainbow function. For the second possibility as $\rho_0 = \left(\frac{16}{9}\chi\right)^{-2}$, with respect to the positivity of the spatial curvature due to the positive energy condition through equation (31), we need that the barotropic EOS parameter ω satisfies the condition $\omega < -1/3$. Therefore, here we have a stable ESU for the barotropic EOS parameter $\omega < -1/3$, denoting the phantom matter fields, with the bounded energy density of ESU ρ_0 as $\rho_0 < l_{pl}^{-4}$. As mentioned before, the existence of such constraint on energy density of ESU may be absent in other modified gravity theories. Here, such a bound on initial density ρ_0 is arising from the energy dependent metric in gravity's rainbow proposal which is one of the common cut offs of quantum gravity theories. Finally, it should be mentioned that choosing the rainbow function (46) leads to the same result with GR so that a closed universe filled by usual non-relativistic matter fields, is not stable against small linear scalar perturbations.

3.2 In the Presence of Cosmological Constant Λ

In this subsection, we want to consider the possible modification by the cosmological constant $\Lambda(\epsilon)$ and investigate its effects on stability condition of an ESU. In presence of cosmological constant $\Lambda(\epsilon)$, the second modified FRW equation (24) remain without change. But, the first modified FRW equation (23) and the conservation law (25), take the form of

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \frac{1}{f^2} = \frac{8\pi G\rho}{3} \frac{1}{f^2} + \frac{\Lambda(\epsilon)}{3f^2}, \quad (56)$$

and

$$\dot{\rho} + 3\rho\frac{\dot{a}}{a}(1+\omega) = -\frac{\dot{\Lambda}(\epsilon)}{8\pi G}. \quad (57)$$

One may consider another rainbow function as $h(\epsilon)$ so that $\Lambda(\epsilon) = h(\epsilon)^2\Lambda$ where Λ is the usual cosmological constant. By this consideration, the modified FRW equation (56) will be

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \frac{1}{f^2} = \frac{8\pi G\rho}{3} \frac{1}{f^2} + \frac{h(\epsilon)^2\Lambda}{3f^2}. \quad (58)$$

By setting this new rainbow function as $h(\epsilon)^2 = (1+\lambda\rho)$, see second Ref. of [27], for the ESU described by $\dot{a} = \ddot{a} = 0$, we get the following equation from equation (56) as

$$\frac{k}{a_0^2} = \frac{8\pi G\rho_0}{3} + \frac{\Lambda}{3}(1+\lambda\rho_0), \quad (59)$$

where λ is a dimensional parameter. By keeping positive energy condition, the above equation implies that positive spatial curvature $k > 0$, for two cases $\Lambda > 0$ and $\Lambda < 0$, is guaranteed only under the following constraints, respectively

$$\lambda > -\frac{1}{\rho_\Lambda} - \frac{1}{\rho_0}, \quad (60)$$

and

$$\lambda < \frac{1}{|\rho_\Lambda|} - \frac{1}{\rho_0}, \quad (61)$$

so that $\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}$ and $|\rho_\Lambda| \equiv \frac{|\Lambda|}{8\pi G}$. With note to the our previous result on upper bounds on energy density of ESU as $\rho_0 < l_{pl}^{-4}$, the above constraints reads as follows

$$\lambda > -\frac{1}{\rho_\Lambda} - l_{pl}^4, \quad (62)$$

and

$$\lambda < \frac{1}{|\rho_\Lambda|} - l_{pl}^4. \quad (63)$$

In DGR formalism of gravity, it is expected that parameter λ to have the order of magnitude $|\lambda| \sim l_{pl}^4$. As it seems, this value of λ be satisfied both the above constraints. Also, in the presence of cosmological constant, we recover Eq. (32) from the second modified FRW equation (24). Similar to the previous analysis, by inserting equation (33) into the equation (56) and neglecting the non-linear perturbation terms, we get

$$-\frac{2k}{a_0^2}\delta a = \left(\frac{8\pi G}{3} + \frac{\Lambda}{3}\lambda\right)\rho_0\delta\rho \quad (64)$$

It is seen that for the case of positive spatial curvature $k > 0$, the sing of variation of the scale factor δa is in contrast to the variation of matter density $\delta\rho$, provided that

$$\lambda > -\frac{1}{\rho_\Lambda}, \quad (65)$$

and

$$\lambda < \frac{1}{|\rho_\Lambda|}, \quad (66)$$

where are related to the cases $\Lambda > 0$ and $\Lambda < 0$, respectively. For the case of positive cosmological constant, $\Lambda > 0$, by comparing the constraints (60) and (65), one realizes that constraint (60) can be satisfied via constraint (65) while its inverse is not true and so relation (65) will be a tighter constraint. With the same reason, one can say that for the case of negative cosmological constant, $\Lambda < 0$, constraint (61) is more tighter than (66). Finally, by using equation (64) and considering equation (33) into the second modified FRW equation (24), after neglecting the non-linear perturbation terms, we arrive to following differential equation

$$\delta\ddot{a} - \frac{k}{a_0^2} \left(\frac{1}{1 + \lambda\rho_\Lambda}\right) \left(1 - \frac{10}{3}\xi\rho_0^{\frac{1}{3}}\right) (1 + 3\omega)\delta a = 0. \quad (67)$$

By putting $\rho_0 = \left(\frac{4}{3}\xi\right)^{-4}$, the above differential equation takes the form of

$$\delta\ddot{a} + \frac{3k}{2a_0^2} \left(\frac{1}{1 + \lambda\rho_\Lambda}\right) (1 + 3\omega)\delta a = 0, \quad (68)$$

In order to have a stable ESU against homogeneous and linear scalar perturbation described by the oscillatory modes, we have two possibilities as

- For the case of $\Lambda > 0$

$$\omega > -\frac{1}{3}, \quad \lambda > -\frac{1}{\rho_\Lambda}, \quad \text{or} \quad \omega < -\frac{1}{3}, \quad \lambda < -\frac{1}{\rho_\Lambda}. \quad (69)$$

- For the case of $\Lambda < 0$

$$\omega > -\frac{1}{3}, \quad \lambda < \frac{1}{|\rho_\Lambda|}, \quad \text{or} \quad \omega < -\frac{1}{3}, \quad \lambda > \frac{1}{|\rho_\Lambda|}. \quad (70)$$

Now, in the same way as above by using the rainbow function (27), by choosing the rainbow function (46), we obtain the following differential equation

$$\delta\ddot{a} - \frac{k}{a_0^2} \left(\frac{1}{1 + \lambda\rho_\Lambda} \right) \left(1 + \frac{8}{3}\chi\rho_0^{\frac{1}{2}} \right) (1 + 3\omega)\delta a = 0, \quad (71)$$

where by inserting $\rho_0 = \left(\frac{16}{9}\chi\right)^{-2}$, coming from constraint (32) for the rainbow function (46), it can be rewritten as

$$\delta\ddot{a} - \frac{5k}{2a_0^2} \left(\frac{1}{1 + \lambda\rho_\Lambda} \right) (1 + 3\omega)\delta a = 0. \quad (72)$$

This differential equation possesses the stable oscillatory modes under the following possibilities

- The case of $\Lambda > 0$

$$\omega > -\frac{1}{3}, \quad \lambda < -\frac{1}{\rho_\Lambda}, \quad \text{or} \quad \omega < -\frac{1}{3}, \quad \lambda > -\frac{1}{\rho_\Lambda}. \quad (73)$$

- The case of $\Lambda < 0$

$$\omega > -\frac{1}{3}, \quad \lambda > \frac{1}{|\rho_\Lambda|}, \quad \text{or} \quad \omega < -\frac{1}{3}, \quad \lambda < \frac{1}{|\rho_\Lambda|}. \quad (74)$$

We note that since the cosmological constant does not appear in constraint equation (32), the energy density of ESU is bounded as $\rho_0 < \epsilon_{pl}^4$ or $\rho_0 < l_{pl}^{-4}$, for the both rainbow functions (27) and (46). It is found that in order to have stable solutions for an ESU in the presence of a cosmological constant, beside the cutoff on energy density ρ_0 , the conditions (69), (70) and (73), (74) for each of rainbow functions (27) and (46), should also be satisfied respectively. For the case of positive cosmological constant $\Lambda > 0$, by comparing conditions mentioned in (69) and (73) with the constraint (65), one realize that with choice of rainbow functions (27) and (46), the solutions with $\omega > -\frac{1}{3}$, $\lambda > -\frac{1}{\rho_\Lambda}$ and $\omega < -\frac{1}{3}$, $\lambda > -\frac{1}{\rho_\Lambda}$ are allowed, respectively. For the case of negative cosmological constant $\Lambda < 0$, constraint (61), is more tighter than (66). In fact, we recall that the constraint (61) is important in the sense that with respect to physical energy condition in quantum gravity regimes, ensures that $k > 0$. So, it is clear that all the constraints on the parameter λ in (70) and (74) violate the constraint (61). Overall, this means that in the presence of a quantum gravity modifications such as what is done by gravity's rainbow, with respect to positive energy condition and positive spatial curvature $k > 0$, which is one of the basic assumptions in modern version of emergent universe scenario, for the cosmological models possessing a negative cosmological constant, there is no stability for an ESU.

4 Einstein Static Universe, Vector and Tensor Perturbations

In the cosmological context, the vector perturbations of a perfect fluid having energy density ρ and barotropic pressure $p = \omega\rho$ are governed by the co-moving dimensionless *vorticity* defined as $\varpi_a = a\varpi$. The vorticity modes satisfy the following propagation equation [37]

$$\dot{\varpi}_\kappa + (1 - 3c_s^2)H\varpi_\kappa = 0, \quad (75)$$

where $c_s^2 = dp/d\rho$ and H are the sound speed and the Hubble parameter, respectively. This equation is valid in our treatment of Einstein static Universe in the framework of the rainbow gravity through the modified Friedmann equations (23) and (24). For the Einstein static Universe with $H = 0$, the equation (75) reduces to

$$\dot{\varpi}_\kappa = 0, \quad (76)$$

where indicates that the initial vector perturbations remain frozen and consequently we have neutral stability against the vector perturbations. Tensor perturbations, namely gravitational-wave perturbations, of a perfect fluid is described by the co-moving dimensionless transverse-traceless shear tensor $\Sigma_{ab} = a\sigma_{ab}$, whose modes satisfy the following equation

$$\ddot{\Sigma}_\kappa + 3H\dot{\Sigma}_\kappa + \left[\frac{\mathcal{K}^2}{a^2} + \frac{2k}{a^2} - \frac{(1+3\omega)\rho - 2\Lambda(\epsilon)}{3} \right] \Sigma_\kappa = 0, \quad (77)$$

where \mathcal{K} is the co-moving index ($D^2 \rightarrow -\mathcal{K}^2/a^2$ in which D^2 is the covariant spatial Laplacian)[37]. For the Einstein static Universe ($H = 0$), this equation reduces to

$$\ddot{\Sigma}_\kappa + \left[\frac{\mathcal{K}^2}{a_0^2} + \frac{2k}{a_0^2} - \frac{(1+3\omega)\rho_0 - 2\Lambda(1+\lambda\rho_0)}{3} \right] \Sigma_\kappa = 0. \quad (78)$$

Note that in the general tensor perturbation equation (77), the Hubble parameter H contains the rainbow function f through equation (23). But, because for the Einstein static universe we have $H = 0$, then the rainbow function f did not appear in the equation (78). Then, in order to have stable modes against the tensor perturbations, the following inequality should be satisfied

$$\frac{\mathcal{K}^2}{a_0^2} + \frac{2k}{a_0^2} > \frac{(1+3\omega)\rho_0 - 2\Lambda(1+\lambda\rho_0)}{3}. \quad (79)$$

Where, for a closed universe with $k = 1$, considering the eigenvalue spectra $\mathcal{K}^2 = n(n+2)$ with $n = 1, 2, 3, \dots$ [38], will takes the form of

$$\frac{n^2 + 2n + 2}{a_0^2} > \frac{(1+3\omega)\rho_0 - 2\Lambda(1+\lambda\rho_0)}{3}. \quad (80)$$

This inequality gives a restriction on the scale factor of an ESU in terms of its matter density and background cosmological constant. Even though the existence of open and flat solutions are possible in some modified gravity models, in the present modified gravity model it is forbidden by regarding the weak energy condition. Therefore, the stability analysis in the present model is restricted to the physically viable closed cosmological model. For the the eigenvalue spectra \mathcal{K}^2 for open and flat models, one is referred to [38].

5 Summery and Conclusion

According to a modern version of emergent cosmological scenario proposed by Ellis and Maartens [16], the early Universe before passing to the inflationary phase, has experienced an eternal Einstein static state rather than a Big-Bang singularity. More exactly, it refers to a static closed space in the asymptotic past (before entering the Universe to the period of inflation) known as “Einstein static Universe” (ESU). The initial conditions as the quantum gravity effects in the early high energy Universe, will influence the stability of this static state. For this reason, in the present work, we have examined the effects of an effective approach to quantum gravity proposed by Magueijo and Smolin [10], known as “gravity’s rainbow” or “doubly general relativity (DGR)”, on the stability of the Einstein static state against the linear homogeneous scalar, vector and tensor perturbations. In order to following our aim in the framework of DGR, we have needed to introduce appropriate rainbow functions. Then, we have considered two appropriate well known rainbow functions (27) and (46). First, by following the phase space mechanism represented in [31], we have investigated the fixed points belonging to a flat FRW model which is modified by the presence of the rainbow function (27). Indeed, these fixed points are de Sitter space solutions of typical flat rainbow FRW model. Our results indicate that although the needed time to get a fixed point is infinite, it may not lead to the elimination of the initial finite time singularities (Big-Bang). This is because of the fact that the fixed points corresponding to past

singularities (Big-Bang) are unstable and may be collapse. Then, we studied the DGR theory in the context of the emergent universe scenario and tried to find its stable Einstein static universe with the required conditions. In order to find a stable ESU, as the first result, we realized that similar to standard GR scenario and independent of the choice of rainbow function, the positive spatial curvature $k > 0$, is the only option respecting to positive energy condition $\rho_0 > 0$ for an ESU in the framework of the DGR theory. For the case of zero cosmological constant $\Lambda = 0$, we found that in order to achieve a stable ESU against homogeneous and linear scalar perturbations, we need a conditions on the energy density of ESU and the barotropic equation of state as $\rho_0 = (\frac{4}{3}\xi)^{-4}$ and $\omega > -\frac{1}{3}$, respectively. Similarly, for the rainbow function (46), we also verified that Big-Bang singularity, can still exists. Unlike the previous case, the time needed to get the fixed point is not infinite in this case. Furthermore, the only fixed point which is revealed in the presence of rainbow function (46), is a future fixed point and is not a past fixed point. Then, by looking at the DGR theory from the emergent universe point of view, we studied the stability of the Einstein static universe and its required conditions. For this case, a stable solution for ESU is guaranteed under these conditions $\rho_0 = (\frac{16}{9}\chi)^{-2}$ and $\omega < -\frac{1}{3}$ describing the exotic matter fields. As it is seen, stability conditions $\rho_0 = (\frac{4}{3}\xi)^{-4}$ and $\rho_0 = (\frac{16}{9}\chi)^{-2}$ are direct result from the idea of an energy dependent metric in the framework of gravity's rainbow scenario. These results are equivalent to an explicit cutoff on the energy density of an ESU as $\rho_0 < \epsilon_{pl}^4$ and $\rho_0 < l_{pl}^4$, respectively. Given that, ESU point out a initial static state (or static closed space) of universe before getting into inflationary phase, so the existence of such explicit cutoff (or upper bounds) on ρ_0 could be interpreted as a result of initial dominate quantum gravity effects such as “gravity's rainbow”. In the following of our analysis, we take an energy dependent cosmological constant via the introduction of new rainbow function as $h(\epsilon)^2 = (1 + \lambda\rho)$. It is seen that the positive spatial curvature, $k > 0$, dictates an opposite sing for δa relative to energy density $\delta\rho$ through the equation (64). We find that in order to have stable solutions for an ESU, beside the cutoff on energy density ρ_0 , the conditions (69), (70) and (73), (74) for each of rainbow functions (27) and (46), should also be satisfied respectively. In particular, for the case of positive cosmological constant $\Lambda > 0$, by comparing conditions mentioned in (69) and (73) with the constraint (65), one realize that with choice of rainbow functions (27) and (46), the solutions with $\omega > -\frac{1}{3}$, $\lambda > -\frac{1}{\rho_\Lambda}$ and $\omega < -\frac{1}{3}$, $\lambda > -\frac{1}{\rho_\Lambda}$ are allowed, respectively. For the case of negative cosmological constant $\Lambda < 0$, constraint (61), is more tighter than (66). In fact, we recall that the constraint (61) is important in the sense that with respect to physical energy condition in quantum gravity regimes, ensures that $k > 0$. So, it is clear that all the constraints on the parameter λ in (70) and (74) violate the constraint (61). Overall, this means that in the presence of a quantum gravity modifications such as what is done by gravity's rainbow, with respect to positive energy condition and positive spatial curvature $k > 0$, which is one of the basic assumptions in modern version of emergent universe scenario, for cosmological models with a negative cosmological constant, there is no stability for an ESU. Finally, we investigated the the stability of an ESU in the framework of DGR versus vector and tensor perturbations. It is found that there is a neutral stability against the vector perturbations. In order to have the stability against the tensor perturbations, the scale factor of an ESU is restricted by its matter density and background cosmological constant.

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